Ex 1 Let $f(x) = e^x$. Find the degree k polynomial, $g_k(x)$, which has $g_k(0) = f(0)$, $g_k'(0) = f'(0)$, $g_k''(0) = f''(0)$,..., $g_k^{(k)}(0) = f^{(k)}(0)$.

$g_0(x) = a_0$, need $g_0(0) = f(0) = e^0 = 1 \longrightarrow g_0(x) = 1$

$g_1(x) = a_0 + a_1 x$

- need $g_1(0) = f(0) = e^0 = 1 \longrightarrow a_0 = 1$

  $g_1(0) = a_0$

- need $g_1'(0) = f'(0) = e^0 = 1 \longrightarrow a_1 = 1$

  $g_1'(0) = a_1$

$\longrightarrow g_1(x) = 1 + x$

$g_2(x) = 1 + x + \frac{x^2}{2}$

$g_k(x) = 1 + x + \frac{x^2}{2} + \dots + \frac{x^k}{k!}$

* space

Ex 2 Let $f(x) = e^x$. Find the degree k polynomial, $h_k(x)$, which has $h_k(5) = f(5)$, $h_k'(5) = f'(5)$, $h_k''(5) = f''(5)$,..., $h_k^{(k)}(5) = f^{(k)}(5)$.

$h_0(x) = b_0 \longrightarrow$ need $b_0 = h_0(5) = f(5) = e^5 \longrightarrow h_0(x) = e^5$

$h_1(x) = mx + b$

↳ $5m + b = h_1(5) = f(5) = e^5$

↳ $m = h_1'(5) = f'(5) = e^5$

$\longrightarrow 5e^5 + b = e^5$

$\longrightarrow b = -4e^5$

$\longrightarrow h_1(x) = e^5 x - 4e^5$

Problems: solving equations, $b \neq b_0$

Alternate approach: $h_1(x) = b_0 + b_1(x - 5)$

↳ $b_0 = h_1(5) = f(5) = e^5$

↳ $b_1 = h_1'(5) = f'(5) = e^5$

$\longrightarrow h_1(x) = e^5 + e^5(x - 5)$

$h_2(x)$ ...

Q. What if we take k to infinity?

Note that $1 + x + \frac{x^2}{2} + \cdots + \frac{x^k}{k!}$ is the kth partial sum of $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

So for x "near" 0, $\sum_{n=0}^{\infty} \frac{x^n}{n!} \approx e^x$

For x "near" 5, $\sum_{n=0}^{\infty} \frac{e^5 \cdot (x-5)^n}{n!} \approx e^x$

Questions:

(1) For which x does $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

(2) If $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges, what does it converge to?

Def: A series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$ is called a power series

Q. How do you want to think about power series?

① As a function of x

② Fix x, think about as a number

Key Skill:

① See x as a variable when
- ↳ deriving
- ↳ integrating

② See x as a constant when
- ↳ computing values
- ↳ determing convergence

Def: For the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$...

- The constant "a" is the center
- The numbers $c_0, c_1, c_2, \ldots$ are the coefficients

Q. For which values of x does a power series converge?

↳ Think about x as a number

Ex 3 For which values of x does the power series $\sum_{n=0}^{\infty} x^n$ converge?

Geometric (x replaces r)

$\sum_{n=0}^{\infty} x^n$ converges $\iff |x| < 1 \iff x \in (-1, 1)$

Ex 4 For which values of x does the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

Ratio Test: $\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \cdot \frac{n!}{(n+1)!} \right|$

$= \lim_{n\to\infty} \left| x \cdot \frac{1}{n+1} \right| = 0 < 1$ ← no matter what x is

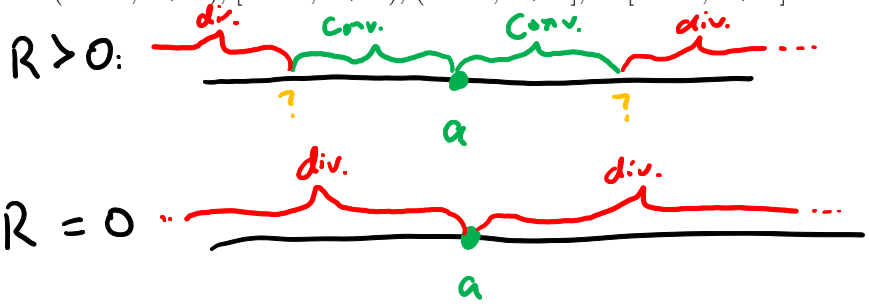
$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x $\iff x \in (-\infty, \infty)$

Ex 5 For which values of x does the power series $\sum_{n=0}^{\infty} n! \cdot x^n$ converge?

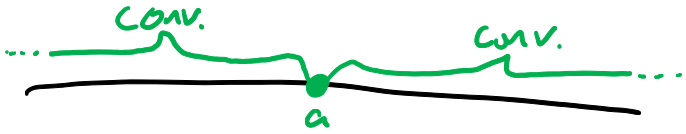
R.T.

Thm: If I is the set of all x-values for which $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges, there are two possibilities.

(a) The series converges on an interval of "radius" $R \geqslant 0$, in which case I is one of the following: $(a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$, or $[a-R, a+R]$.



(b) The series converges for all x, in which case $I = (-\infty, \infty)$.



Def: In the setting of the above theorem,

- I is called the interval of convergence
- R is called the radius of convergence

Q. What is R if $I = (-\infty, \infty)$?

We say $R = \infty$

Q. Given a power series, how do we find the radius and interval of convergence?

Use the root or ratio test

Ex 6 Find the center, radius of convergence, and interval of convergence for $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}5^n}(x-3)^n$.

The center is...

(a) -5 (c) -1 (e) 0.5 (g) 3
(b) -2 (d) 0 (f) 1 (h) 5

Goal: $1 > \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{\sqrt{n+1}\cdot 5^{n+1}} \cdot \frac{\sqrt{n}\cdot 5^n}{(x-3)^n}\right|$

$= \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{(x-3)^n} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \cdot \frac{5^n}{5^{n+1}}\right| = \lim_{n\to\infty} \left|(x-3)\cdot\sqrt{\frac{n}{n+1}}\cdot\frac{1}{5}\right|$

$= \frac{|x-3|}{5}$

$1 > \frac{|x-3|}{5} \iff 5 > |x-3| \iff x$ is within 5 of 3

$\iff x \in (-2, 8)$

Note $x \in (-2,8)$ implies that $\sum_{n=1}^{\infty} \frac{(x-3)^n}{\sqrt{n}\cdot 5^n}$ converges

but there might be more x-values.

Fact: the only other candidates are the endpoints: -2 and 8

Check: $x = 8$

↳ $\sum_{n=1}^{\infty} \frac{(8-3)^n}{\sqrt{n}\cdot 5^n} = \sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}5^n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the p-test ($p=\frac{1}{2}$)

Check: $x = -2$

↳ $\sum_{n=1}^{\infty} \frac{(-2-3)^n}{\sqrt{n}\cdot 5^n} = \sum_{n=1}^{\infty} \frac{(-5)^n}{\sqrt{n}\cdot 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the A.S.T.

$\Rightarrow I = [-2, 8)$, $R = 5$

Q. How do we find I in general?

(a) Compute the limit from the root or ratio test.

(b) Find the interval of x-values that make the limit less than 1.

(c) Check the endpoints using the ch. 5 convergence tests

Ex 7 Find the interval of convergence for $\sum_{n=0}^{\infty} n^n(x-3)^n$.

Goal: $1 > \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} |n(x-3)|$

$$= \begin{cases} \infty & x \neq 3 \\ 0 & x = 3 \end{cases}$$

$I = [3, 3]$

$R = 0$

Ex 8 Find the interval of convergence for $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$. The center is...

(a) -1 (b) 0 (c) 1 (d) 2

$\frac{x^{2k+1}}{(-3)^k(2k+1)}$

Goal: $1 > \lim_{k\to\infty} \left| \frac{x^{2(k+1)+1}}{(-3)^{k+1}(2(k+1)+1)} \cdot \frac{(-3)^k(2k+1)}{x^{2k+1}} \right|$

$= \lim_{k\to\infty} \left| \frac{x^{2k+3}}{x^{2k+1}} \cdot \frac{(-3)^k}{(-3)^{k+1}} \cdot \frac{2k+1}{2k+3} \right|$

$= \lim_{k\to\infty} \left| x^2 \cdot \frac{1}{-3} \cdot \frac{2k+1}{2k+3} \right|$

where's the lie?

$= x^2 \cdot \frac{1}{3}$

$1 > \frac{x^2}{3} \longleftrightarrow 3 > x^2 \longleftrightarrow \sqrt{3} > |x| \longleftrightarrow x \in (-\sqrt{3}, \sqrt{3})$

Check: $x = -\sqrt{3}$

$\sum_{k=0}^{\infty} \frac{(-\sqrt{3})^{2k+1}}{(-3)^k(2k+1)} = \sum_{k=0}^{\infty} \frac{(-\sqrt{3})^{2k}\cdot(-\sqrt{3})}{(-3)^k(2k+1)} = \sum_{k=0}^{\infty} \frac{\left((-\sqrt{3})^2\right)^k(-\sqrt{3})}{(-3)^k\cdot(2k+1)}$

$= \sum_{k=0}^{\infty} \frac{3^k}{(-3)^k} \cdot \frac{-\sqrt{3}}{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k(-\sqrt{3})}{2k+1}$

converges by the A.S.T.

Check: $x = \sqrt{3}$

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Ex 9 Find the interval of convergence for $\sum_{n=0}^{\infty} \frac{(2x+8)^n}{n!}$. The center is...

(a) -8 (c) -2 (e) 2 (g) 8
(b) -4 (d) 0 (f) 4

Move Down

Goal: $1 > \lim_{n\to\infty} \left| \frac{(2x+8)^{n+1}}{(n+1)!} \cdot \frac{n!}{(2x+8)^n} \right| = \cdots = 0$

Note: Not all power series look like $\sum_{n=0}^{\infty} a_n(x-a)^n$ right away. We want to be able to make them look like that

Ex 10 Find the appropriate center a and coefficients $(c_0, c_1, c_2, \dots)$ such that $\sum_{m=0}^{\infty} m!x^{2m} = \sum_{n=0}^{\infty} c_n(x-a)^n$.

a_n

$$\sum_{m=0}^{\infty} m!\,x^{2m} = 1 + x^2 + 2x^4 + 6x^6 + 24x^8 + \cdots + m!x^{2m} + \cdots$$

$$= 1 + 0x + x^2 + 0x^3 + 2x^4 + 0x^5 + \cdots + c_n x^n + \cdots$$

The thing being factorialed is half of the exponent

$a = 0$

$$c_n = \begin{cases} 0 & \text{odd } n \\ \left(\frac{n}{2}\right)! & \text{even } n \end{cases}$$

↳ exponent

Ex 11 Find the appropriate center a and coefficients $(c_0, c_1, c_2, \dots)$ such that $\sum_{n=0}^{\infty} \frac{(2x+8)^n}{n!} = \sum_{n=0}^{\infty} c_n(x-a)^n$.

$$\sum_{n=0}^{\infty} \frac{(2x+8)^n}{n!} = \sum_{n=0}^{\infty} \frac{(2(x+4))^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n \cdot (x+4)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{n!}\left(x-(-4)\right)^n$$

$a = -4$

$c_n = \frac{2^n}{n!}$

Q. How can we tell what a power series converges to?

We can't (in general)

Q. How can we convert familiar functions to power series?

① Make it look like something we know how to handle

② Something else???

Ex 12 Find the power series centered at $a = 9$ which converges to $f(x) = \frac{2}{1-x}$ and identify the interval of convergence of the power series.

Want to force $x-9$ to show up

$$f(x) = \frac{2}{1-x} = \frac{2}{1-x-9+9} = \frac{2}{1-(x-9)-9} = \frac{2}{-8-(x-9)}$$

$$= \frac{2}{-8} \cdot \frac{1}{1+\left(\frac{x-9}{8}\right)} = -\frac{1}{4} \cdot \frac{1}{1-\left(\frac{x-9}{-8}\right)} = -\frac{1}{4} \cdot \frac{1}{1-u}$$

$$u = \frac{x-9}{-8}$$

$$= -\frac{1}{4} \cdot \sum_{n=0}^{\infty} u^n = \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)\left(\frac{x-9}{-8}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{-1}{4 \cdot (-8)^n} \cdot (x-9)^n$$

Converges iff. $|u| < 1 \iff \left|\frac{x-9}{-8}\right| < 1$

$\iff |x-9| < 8$

$\iff x \in (1, 17) = I$

Ex 13 Find the power series for $f(x) = \frac{x}{4+x^2}$ centered at $a = 0$ and identify the interval of convergence.

Q. What's a helpful form for $f(x)$?

(a) $f(x) = x \cdot \dfrac{1}{4+x^2}$

(b) $f(x) = \dfrac{x}{4} \cdot \dfrac{1}{1+\frac{x^2}{4}}$

(c) $f(x) = \dfrac{x}{-4} \cdot \dfrac{1}{-1-\frac{x^2}{4}}$

(d) $f(x) = \dfrac{x}{4} \cdot \dfrac{1}{1-(-\frac{x^2}{4})}$

$$f(x) = \frac{x}{4} \cdot \frac{1}{1-\left(-\frac{x^2}{4}\right)} = \frac{x}{4} \cdot \sum_{n=0}^{\infty} \left(-\frac{x^2}{4}\right)^n$$

$$= \frac{x}{4} \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{4^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{4^{n+1}}$$

Converges iff. $\left|\frac{-x^2}{4}\right| < 1 \iff x^2 < 4 \iff |x| < 2$

$I = (-2, 2)$

Ex 14 Find the power series for $f(x) = \frac{1}{x}$ centered at $a = 1$ and identify the interval of convergence.

Q. What's the most helpful form for $f(x)$?

(a) $f(x) = \dfrac{1}{x}$

(b) $f(x) = \dfrac{1}{1-(1-x)}$

(c) $f(x) = \dfrac{1}{1-(-(x-1))}$

(d) $f(x) = x^{-1}$

$$f(x) = \frac{1}{1-(-(x-1))} = \sum_{n=0}^{\infty} (-(x-1))^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

Converges iff. $|-(x-1)| < 1 \iff |x-1| < 1$

$\iff x \in (0,2) = I$

Ex 15 Find the power series for $f(x) = \dfrac{9-3x}{x^2-6x}$ centered at $a = 3$.

Q. What technique do you think will be helpful here?

(a) Factor the denominator
(b) Partial fraction decomposition
(c) Complete the square in the denominator
(d) Long divide the numerator and the denominator

$$f(x) = \frac{-3(x-3)}{x^2-6x+9-9} = \frac{-3(x-3)}{(x-3)^2-9}$$

$$= \frac{-3(x-3)}{-9} \cdot \frac{1}{1-\frac{(x-3)^2}{9}}$$

$$= \frac{(x-3)}{3} \cdot \sum_{n=0}^{\infty} \left(\frac{(x-3)^2}{3^2}\right)^n$$

$$= \frac{(x-3)}{3} \sum_{n=0}^{\infty} \frac{(x-3)^{2n}}{3^{2n}} = \sum_{n=0}^{\infty} \frac{(x-3)^{2n+1}}{3^{2n+1}}$$

Converges iff. $\left|\dfrac{(x-3)^2}{9}\right| < 1 \iff (x-3)^2 < 9$

$\iff |x-3| < 9$

$\iff x \in (6, 12) = I$